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Research Article

On ideals and congruences in BBG-algebras

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Abstract

In this paper, we define the notions of ideals and congruences on BBG-algebras and investigate some related properties. Moreover, we study the relation between ideals and congruences on commutative BBG-algebras. We define the quotient in commutative BBG-algebras induced by this relation and study its properties.

Keywords: BBG-algebras, congruences, ideals, quotient BBG-algebras

Introduction

In 1998, Dudek and Zhang introduced a new concept of ideals in BCC-algebras and described connections between such ideals and congruences. (Dudek & Zhang, 1998) In 2007, Ding and Pang introduced on the properties of the element and then the definition and properties of congruences and quotient algebras are given and quotient algebras are the basic tools for explored the structures of BCI-algebras. (Ding & Pang, 2007) In 2009, Prabpayak and Leerawat introduced some kind of algebras which is called KU-algebras and defined ideals and studied congruences on KU-algebras. Furthermore, they investigated some of its properties. (Prabpayak & Leerawat, 2009) In 2010, Asawasamrit and Leerawat introduced an algebraic stucture called a binary algebra and studied the relation between ideals and congruences. Finally, they defined quotient binary algebra and studied its properties. (Asawasamrit & Leerawat, 2010) In 2012, Asawasamrit and Sudprasert introduced to the genral theory of KK-algebras and shown relation between ideals and congruences to defined qutient KK-algebra and investigated fundamental properties. (Asawasamrit & Sudprasert, 2012)

In 2015, Tearnbucha and Asawasamrit introduced a new algebraic structure called a BBG-algebra and studied its fundamental properties (Tearnbucha & Asawasamrit, 2015). They defined a BBG-algebra as an algebra $(X, *, 0)$ with a binary operation $*$ which satisfies the following properties:

(BBG-1) $x * x = 0$,
(BBG-2) $0 * x = x$,
(BBG-3) $x * (y * z) = ((y * 0) * x) * z$, for all $x, y, z \in X$.

Before proved the basic properties of BBG-algebras, they gave some examples including the following (Tearnbucha & Asawasamrit, 2015).

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Example 1.1 Let $X = \{0, 1, 2\}$ and $*$ be defined by the Cayley Table 1. Then they showed by a straightforward, but lengthy, calculation that $(X, *, 0)$ satisfied the three axioms of a BBG-algebra.

Table 1. Definition of $*$ for example 1.1 with $X = \{0, 1, 2\}$.

$*$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Proposition 1.2 Let $(X, *, 0)$ be a BBG-algebra. Then, for all $x, y, z \in X$:

1. $(x * 0) * (x * y) = y$,
2. $x * y = x * z$ implies $y = z$,
3. $(x * y) * z = y * ((x * 0) * z)$,
4. $x * y = 0$ implies $x = y$,
5. $(x * 0) * 0 = x$,
6. $(y * x) * 0 = x * y$,
7. $y * x = z * x$ implies $y = z$.

Tearnbucha and Asawasamrit defined a commutative BBG-algebra $(X,*,0)$ as follows (Tearnbucha & Asawasamrit, 2015).

Definition 1.3 A commutative BBG-algebra is a BBG-algebra with the property that $(x * 0) * y = (y * 0) * x$ for all $x, y, z \in X$.

They then defined a binary operation $\wedge$ on a commutative BBG-algebra $(X, *, 0)$ by $x \wedge y = (x * y) * y$ and showed that $x \wedge y = y \wedge x$ for all $x, y \in X$ and proved the following proposition.

Proposition 1.4 If $(X ,*, 0)$ is a commutative BBG-algebra, then for all $a,b, x, y, z \in X$:

1. $(y * 0) * (x * 0) = x * y$,
2. $(x * y) * y = x$,
3. $(y * b) * (x * a) = (a * b) * (x * y)$,
4. $(z * x) * (z * y) = x * y$,
5. $(y * z) * (x * z) = x * y$,
6. $(b * y) * x = (x * y) * b$,
7. $x * (y * z) = y * (x * z)$.

In this paper, we introduce the notion of ideals and congruence in BBG-algebras and investigate some of their properties. Moreover, we will show some of the relationships between ideals and congruences in commutative BBG-algebras to define the quotient in commutative BBG-algebras and study its properties.

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Ideals

Definition 2.1 Let $(X, *, 0)$ be a BBG-algebra and A be a non-empty subset of X. Then, A is called a closed subset of X on condition that $x * y \in A$ whenever $x, y \in A$.

Lemma 2.2 If A is a closed subset of a BBG-algebra $(X, *, 0)$, then $0 \in A$.
Proof. If $x \in A$, then $x * x \in A$. From BBG-1, $x * x = 0$ and therefore $0 \in A$.

Definition 2.3 Let $(X, *, 0)$ be a BBG-algebra and A be a subset of X. Then A is called an ideal of $(X, *, 0)$ under the operation $*$, if it satises the following conditions

(I-1) $0 \in A$.

(I-2) For $x, y \in X$, if $x \in A$ and $x * y \in A$ then $y \in A$.

Example 2.4 Let $X = \{0, 1, 2, 3, 4, 5\}$ and let the binary operation $*$ be defined by Table 2.

Table 2. Definition of $*$ for Example 2.4 with $X = \{0, 1, 2, 3, 4, 5\}$.

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	2	0	1	4	5	3
2	1	2	0	5	3	4
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

Then, it can easily be checked that $(X, *, 0)$ is a BBG-algebra and that $I = \{0, 1, 2\}$, $J = \{0, 3\}$ and $K = \{0, 5\}$ are closed ideals of X under the operation $*$.

Example 2.5 Let $X = \{0, 1, 2, 3\}$ and let the binary operation $*$ be defined by Table 3.

Table 3. Definition of $*$ for Example 2.4 with $X = \{0, 1, 2, 3\}$.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then, it can easily be checked that $(X, *, 0)$ is a commutative BBG-algebra and that $I = \{0, 2\}$ is closed ideals of X under the operation $*$.

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Lemma 2.6 Let $(X, *, 0)$ be a commutative BBG-algebra and let A be a closed subset of X. Then A is an ideal of X if and only if $x \in A$ and $z * y \notin A$ imply that $z * (x * y) \notin A$ for all $x, y, z \in X$.

Proof. Let A be a closed subset of X. Since $0 \in A$, property I-1 of an ideal is satisfied. Now, we prove that A is an ideal only if the stated conditions are true. We prove by contradiction. Suppose that A is an ideal of X for all $x, y, z \in X$ and that $x \in A$ and $z * (x * y) \in A$. By Proposition 1.4(7), we have $z * (x * y) = x * (z * y)$ and therefore from condition I-2 of the definition of an ideal, we have $z * y \in A$. Hence, if $z * y \notin A$ and $z * (x * y) \in A$, then A is not an ideal. Therefore A is an ideal only if $z * y \notin A$ implies that $z * (x * y) \notin A$. We now prove that if the conditions are true then A is an ideal. Assume that A is a closed subset of X, and for all $x, y, z \in X$, we have $x \in A$, and that $z * y \notin A$ implies that $z * (x * y) \notin A$. Let $x \in A$. Then since A is closed, $0 \in A$ and property I-1 of the definition of an ideal is satisfied. We prove property I-2 by contradiction. Assume that A is not an ideal and the conditions are true, i.e., that if $x \in A$ and $z * y \notin A$ then $x * (z * y) \notin A$. If A is not an ideal, then there exists $x \in A$, $x * (z * y) \in A$ and $z * y \notin A$. From the conditions, we have that $z * y \notin A$ implies that $z * (x * y) \notin A$. But, from Proposition 1.4(7), $z * (x * y) = x * (z * y)$ and therefore $x * (z * y) \notin A$ which is a contradiction. Therefore, if the conditions are true then A is an ideal of X. This completes the proof.

Corollary 2.7 Let A be a closed subset of a commutative BBG-algebra X. Then A is an ideal of X if and only if $x \in A$ and $y \notin A$ imply that $x * y \notin A$ for all $x, y \in X$.

Lemma 2.8 Let A be a closed subset of a commutative BBG-algebra X. Then A is an ideal of X if and only if $x * (y * z) \in A$ and $x * z \notin A$ imply $y \notin A$ for all $x, y, z \in X$.

Proof. Let A be a closed subset of X. Then $0 \in A$ and property I-1 of an ideal is satisfied. We now prove that property I-2 of an ideal is satisfied only if the stated conditions are true.
Suppose that for all $x, y, z \in X$ the conditions $x * (y * z) \in A$ and $x * z \notin A$ are satisfied. We assume that A is an ideal and $y \in A$. We now prove that this is a contradiction. From Proposition 1.4(7), $x * (y * z) = y * (x * z) \in A$. Hence, if $y \in A$ and A is an ideal, then from condition *I*-2 of the definition of an ideal, we have $x * z \in A$. This is a contradiction, and therefore if $x * z \notin A$, $y * (x * z) \in A$ and $y \in A$ then A is not an ideal. Therefore A is an ideal only if $x * (y * z) \in A$ implies that $y \notin A$. We now prove that if the stated conditions $x * (y * z) \in A$, $x * z \notin A$ and $y \notin A$ are true, then property I-2 of an ideal is satisfied. We prove property I-2 by contradiction. We assume that the stated conditions are true and that A is not an ideal. Then, if A is not an ideal, there exists $y \in A$, $y * (x * z) \in A$ and $x * z \notin A$. From Proposition 1.4(7), $x * (y * z) = y * (x * z)$ and therefore $y * (x * z) \in A$. But, from the conditions $x * (y * z) \in A$ and $x * z \notin A$ implies that $y \notin A$. This is a contradiction and therefore A is an ideal. This completes the proof.

Corollary 2.9 Let A be a closed subset of a commutative BBG-algebra X. Then A is an ideal of X if and only if $x * y \in A$ and $y \notin A$ implies $x \notin A$ for all $x, y, z \in X$.

Theorem 2.10 Let $\{J_J : i \in \mathbb{N}\}$ be a family of ideals in a BBG-algebra X where $J_n \subseteq J_{n+1}$ for all $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} J_n$ is an ideal of X.

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Proof. We first note that $0 \in \bigcup_{n=1}^{\infty} J_n$. By definition, J_1 is an ideal implies that $0 \in J_1$ and $J_1 \subseteq \bigcup_{n=1}^{\infty} J_n$. To prove property I-2 of an ideal, we will show that $x \in \bigcup_{n=1}^{\infty} J_n$ and $x * y \in \bigcup_{n=1}^{\infty} J_n$ implies that $y \in \bigcup_{n=1}^{\infty} J_n$. Since $x \in \bigcup_{n=1}^{\infty} J_n$ and $x * y \in \bigcup_{n=1}^{\infty} J_n$, it follows that $x \in J_j$ for some $j \in \mathbb{N}$ and $x * y \in J_k$ for some $k \in \mathbb{N}$. Without loss of generality, we can assume that $j \leq k$. Since $J_j \subseteq J_k$, we have $x \in J_k$ and $x * y \in J_k$. Then, since J_k is an ideal, we have from property I-2 of an ideal that $y \in J_k \subseteq \bigcup_{n=1}^{\infty} J_n$. Therefore, property I-2 is also true for $\bigcup_{n=1}^{\infty} J_n$. This completes the proof.

Theorem 2.11 Let $\{J_j : i \in \mathbb{N}\}$ be a family of closed ideals of a BBG-algebra X where $J_n \subseteq J_{n+1}$ for all $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} J_n$ is a closed ideal of X.

Proof. We will first show that $\bigcup_{n=1}^{\infty} J_n$ is a closed subset of X. Let $x, y \in \bigcup_{n=1}^{\infty} J_n$. It follows that $x \in J_j$ for some $j \in \mathbb{N}$ and $y \in J_k$ for some $k \in \mathbb{N}$. Without loss of generality, we can assume that $j \leq k$ and then $J_j \subseteq J_k$. Therefore, $x, y \in J_k$ and since J_k is a closed ideal, we have that $x * y \in J_k \subseteq \bigcup_{n=1}^{\infty} J_n$. This proves that if $x, y \in \bigcup_{n=1}^{\infty} J_n$, then $x * y \in \bigcup_{n=1}^{\infty} J_n$ and therefore $x * y \in \bigcup_{n=1}^{\infty} J_n$ is a closed subset of X. Finally, it follows immediately from Theorem 2.10 that $\bigcup_{n=1}^{\infty} J_n$ is an ideal of X. This completes the proof.

Theorem 2.12 Let $\{ I_j : j \in J \}$ be a family of ideals of a BBG-algebra X. Then $\bigcap_{j \in J} I_j$ is an ideal of X.

Proof. We note first that $0 \in \bigcap_{j \in J} I_j$ because $0 \in I_j$ for all ideals $I_j \subseteq X$.

We now prove property I-2. Let $x \in \bigcap_{j \in J} I_j$ and $x * y \in \bigcap_{j \in J} I_j$. Then, $x \in I_j$ for all j and $x * y \in I_j$ for all j. Since I_1 is an ideal there exists $y_1 \in I_1$ such that $x * y_1 \in I_1$. Similarly, since I_2 is an ideal there exists $y_2 * I_2$ such that $x * y_2 \in I_2$. By Proposition 1.2(2), $x * y_1 = x * y_2$ implies that $y_1 = y_2$ and therefore $y_1 = y_2 \in I_1 \cap I_2$. Therefore, $x \in I_1 \cap I_2$ and $x * y \in I_1 \cap I_2$, implies that there exists $y \in I_1 \cap I_2$. Therefore, $I_1 \cap I_2$ satisfies property I-2 of an ideal. This argument can now be applied to $(I_1 \cap I_2) \cap I_3$ to prove that $(I_1 \cap I_2) \cap I_3$ is an ideal. A repeat of this argument completes the proof.

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Theorem 2.13 Let $\{ I_j : j \in J \}$ be a family of closed ideals of a BBG-algebra X. Then $\bigcap_{j \in J} I_j$ is an closed ideal of X.

Proof. By Theorem 2.12, $\bigcap_{j \in J} I_j$ is an ideal of X. We now show that the intersection of closed subsets of X is a closed subset of X. We first show that the intersection of two closed subsets of X is a closed subset of X. Let $x, y \in I_1 \cap I_2$. Since I_1 is closed, we have $x * y \in I_1$. Similarly, since I_2 is closed, we have $x * y \in I_2$. Therefore, $x * y \in I_1 \cap I_2$. We can now repeat the argument for the two closed ideals $I_1 \cap I_2$ and I_3 to prove that $(I_1 \cap I_2) \cap I_3$ is a closed subset of X. A repeat of this argument completes the proof.

Quotient BBG-algebras

In this section, we study congruence and quotient on BBG-algebras.

Definition 3.1 Let I be an ideal of a BBG-algebra X. Define a relation $\sim$ on X by

$$x \sim y \text{ iff } x * y \in I \text{ and } y * x \in I.$$

Theorem 3.2 If I is an ideal of a commutative BBG-algebra X, then the relation $\sim$ is an equivalence relation on X.

Proof. Let I be an ideal of a commutative BBG-algebra X and $x, y, z \in X$. By BBG-1, $x * x = 0$ and since I is an ideal $x * x = 0 \in I$. That is $x \sim x$. Hence $\sim$ is reflexive. Next, suppose that $x \sim y$. It follows that $x * y \in I$ and $y * x \in I$. Then $y \sim x$, so $\sim$ is symmetric. Finally, we prove the transitivity of $\sim$. Let $x, y, z \in I$ and $x \sim y$ and $y \sim z$. We will show that $x \sim z$. Now, by definition of $\sim$, we have $x * y$, $y * x$, $y * z$, $z * y \in I$. By Proposition 1.3(4), we have $(z * y) * (z * x) = y * x \in I$. However, we know that $z * y \in I$ and therefore since I is an ideal, we have $z * x \in I$. Similarly, using Proposition 1.3(5), we can prove that $x * z \in I$. Thus $\sim$ is transitive. Therefore $\sim$ is an equivalence relation.

Lemma 3.3 Let I be an ideal of a commutative BBG-algebra X. For any $x, y, u, v \in X$, if $u \sim v$ and $x \sim y$, then $u * x \sim v * y$.

Proof. We will first show that $u * x \sim v * y$. Since $u \sim v$, we have $u * v \in I$ and $v * u \in I$. Since X is a commutative BBG-algebra, we have from Proposition 1.3(5) that $(v * x) * (u * x) = u * v \in I$ and $(u * x) * (v * x) = v * u \in I$. Therefore $u * x \sim v * x$. We will next show that $v * x \sim v * y$. Since $x \sim y$, we have $x * y \in I$ and $y * x \in I$. Since X is a commutative BBG-algebra, we have from Proposition 1.3(4) that $(v * x) * (v * y) = x * y \in I$ and $(v * y) * (v * x) = y * x \in I$. Therefore $v * x \sim v * y$. Then, since $u * x \sim v * x$ and $v * x \sim v * y$, we have from transitivity of $\sim$ that $u * x \sim v * y$. This completes the proof.

Corollary 3.4 If I is an ideal of a commutative BBG-algebra X, then the relation $\sim$ is a congruence relation on X.

Proof. For the operation $*$ of the BBG-algebra and for all $x, y, u, v \in X$, we have from Theorem 3.2 that $\sim$ is an equivalence relation on X and from Lemma 3.3 that $u \sim v$ and $x \sim y$ implies that $u * x \sim v * y$. Therefore, $\sim$ is a congruence relation on X.

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Definition 3.5 Let I be an ideal of a commutative BBG-algebra X. Given $x \in X$, the equivalence class $[x]_I$ of x is defined as the set of all elements of X that are equivalent to x, that is,

$$[x]_I = \{y \in X : x * y\}.$$

We define a quotient set by $X/I = \{ [x]_I : x \in X\}$.

Lemma 3.6 Let $x, y \in X/I$. If $[x]_I \in X/I$ and $[y]_I \in X/I$, then $[x * y]_I \in X/I$.

Proof. By definition, if $[x]_I \in X/I$ then $x \in X$ and if $y \in [y]_I$, then $y \in X$. Since $*$ is a binary operation on X, we have $x * y \in X$. Therefore, $[x * y]_I \in X/I$.

Definition 3.7 We define an operation $\circ$ on X/I by

$$[x]_I \circ [y]_I = [x * y]_I.$$

From Lemma 3.6, this operation is a binary operation on X/I.

Theorem 3.8 If I is an ideal of a commutative BBG-algebra X with quotient set $X/I = \{ [x]_I : x \in X\}$ and a binary operation $\circ$, then $\circ$ is a mapping from $X/I \times X/I$ to X/I.

Proof. Let $[x_1]_I, [x_2]_I \in X/I$. From Lemma 3.6, we have $[x_1]_I \circ [x_2]_I \in X/I$.

We prove that if $[y_1]_I = [x_1]_I \circ [x_2]_I \in X/I$ and $[y_2]_I = [x_1]_I \circ [x_2]_I \in X/I$, then $[y_1]_I = [y_2]_I$.

If $[y_1]_I = [x_1]_I \circ [x_2]_I$, then $[y_1]_I = [x_1]_I \circ [x_2]_I$ implies that $y_1 \sim x_1 * x_2$. Similarly, if $[y_2]_I = [x_1]_I \circ [x_2]_I$, then $y_2 \sim x_1 * x_2$. By the transitivity property of $\sim$, we have $y_1 \sim y_2$ and $[y_1]_I = [y_2]_I$. Therefore, if $[x_1]_I \in X/I$ and $[x_2]_I \in X/I$, then there exists a unique $[y]_I = [x_1]_I \circ [x_2]_I \in X/I$. The proof is complete.

Theorem 3.9 Let I be an ideal of a commutative BBG-algebra $(X, *, 0)$. Then $(X/I, \circ, [0]_I)$ is a commutative BBG-algebra. The set X/I is called the quotient commutative BBG-algebra X/I.

Proof. We first prove that $(X/I, \circ, [0]_I)$ is a BBG-algebra.

(BBG-1) If $[x]_I, [y]_I, [z]_I \in X/I$, then $[0]_I \circ [x]_I = [x * x]_I = [0]_I$.

(BBG-2) It is clear that $[0]_I \circ [x]_I = [0 * x]_I = [x]_I$.

(BBG-3) Consider

$$\begin{aligned} [x]_I \circ ([y]_I \circ [z]_I) &= [x]_I \circ [y * z]_I \\ &= [x * (y * z)]_I \\ &= [((y * 0) * x) * z]_I \\ &= [((y * 0) * x]_I \circ [z]_I \\ &= (([y * 0]_I) \circ [x]_I) \circ [z]_I \\ &= (([y]_I \circ [0]_I) \circ [x]_I) \circ [z]_I. \end{aligned}$$

We prove that $(X/I ; \circ, [0]_I)$ is a commutative BBG-algebra.

$$\begin{aligned} ([x]_I \circ [0]_I) \circ [y]_I &= [x * 0]_I \circ [y]_I \\ &= [(x * 0) * y]_I \\ &= [(y * 0) * x]_I \\ &= [y * 0]_I \circ [x]_I \\ &= ([y]_I \circ [0]_I) \circ [x]_I. \end{aligned}$$

Therefore, from Definition 1.3, $(X/I, \circ, [0]_I)$ is a comutative BBG-algebra.

Example 3.10 Let $X = \{0, 1, 2, 3, 4, 5\}$ be a BBG-algebra with $*$ operation defined in Table 2 of Example 2.4. We can define a quotient algebra as $X/I = \{ [0]_I, [3]_I \}$, where $[0]_I = [1]_I$

= $[2]_I$ = { 0, 1, 2 } and $[3]_I = [4]_I = [5]_I$ = {3, 4, 5}. The operator ∘ can be defined on X/I as shown in Table 4.

Table 4: Definition of ∘ for Example 3.10, with $X/I = \{ [0]_I , [3]_I \}$

*	$[0]_I$	$[3]_I$
$[0]_I$	$[0]_I$	$[3]_I$
$[3]_I$	$[3]_I$	$[0]_I$

Example 3.11 Let X = {0, 1, 2, 3} be a BBG-algebra with $*$ operation defined in Table 3 of Example 2.5. We can define a quotient algebra as $X/I = \{ [0]_I , [1]_I \}$, where $[0]_I = [2]_I$ = { 0, 2 } and $[1]_I = [3]_I$ = {1, 3}. The operator ∘ can be defined on X/I as shown in Table 5.

Table 5: Definition of ∘ for Example 3.11, with $X/I = \{ [0]_I , [1]_I \}$

*	$[0]_I$	$[1]_I$
$[0]_I$	$[0]_I$	$[1]_I$
$[1]_I$	$[1]_I$	$[0]_I$

Lemma 3.12 Let I be an ideal of a BBG-algebra X, and J be any subset of X with $I \subseteq J \subseteq X$. Then J is an ideal of X if and only if J/I is an ideal of X/I.
Proof. Let I be an ideal of BBG-algebra X with $I \subseteq J \subseteq X$. We first prove that if J is an ideal of X then J/I is an ideal of X/I. Since I and J are ideals of X, we have from property (I-1) of an ideal that $0 \in I$ and $0 \in J$. Also, since $0 * 0 = 0 \in J$, we have from Definition 3.5 that $[0]_I \in J/I$ and therefore J/I satisfies property (I-1) for an ideal. We now prove property (I-2) for J/I. We assume that $[x]_I \in J/I$ and $[x]_I \circ [y]_I \in J/I$. Therefore, $x \in J$ and also by Definition 3.7, $[x * y]_I = [x]_I \circ [y]_I \in J/I$. Therefore, $x * y \in J$, and since J is an ideal, $x \in J$ and $x * y \in J$ implies that $y \in J$ and hence $[y]_I \in J/I$. We prove that if J/I is an ideal of X/I, then J is an ideal of X. From property (I-1) of an ideal $[0]_I \in J/I$, and therefore $0 \in J$. To prove property (I-2) for J, we must prove that if $x \in J$ and $x * y \in J$ then $y \in J$. We assume that $[x]_I$, $[x]_I \circ [y]_I \in J/I$. Therefore, $x \in J$, since $[x * y]_I = [x]_I \circ [y]_I \in J/I$, we also have $x * y \in J$. However, since J/I is an ideal, $[x]_I$, $[x]_I \circ [y]_I \in J/I$. implies that $[y]_I \in J/I$ and therefore that $y \in J$. Therefore, J satisfies property (I-2) and the proof is complete.

Lemma 3.13 Let I be a closed ideal of a BBG-algebra X and J be a subset of X with $I \subseteq J \subseteq X$. Then J is a closed ideal of X if and only if J/I is a closed ideal of X/I.
Proof. Using Lemma 3.12, we prove that the ideals are closed. We first prove that if J is a closed ideal of X then J/I is a closed ideal of X/I. From Lemma 3.12, J/I is an ideal. Since J is closed, if x, $y \in J$, then $x * y \in J$. Now, assume that $[x]_I \in J/I$ and $[y]_I \in J/I$. Since $[x]_I \circ [y]_I = [x * y]_I$ and $x * y \in J$, we have $[x]_I$, $[x]_I \circ [y]_I \in J/I$ and therefore J/I is closed. We prove that if J/I is a closed ideal of X/I then J is a closed ideal of X. From Lemma 3.12, J is an ideal. Since J/I is closed, if $[x]_I \in J/I$ and $[y]_I \in J/I$, then $[x]_I \circ [y]_I \in J/I$. Therefore, $x \in J$ and $y \in J$. Since $[x * y]_I = [x]_I \circ [y]_I \in J/I$, we have $x * y \in J$. Therefore J is closed and hence J is a closed ideal of X.

Lemma 3.14 If I and J are ideals of a BBG-algebra X with $I \subseteq J \subseteq X$, then I is an ideal of J.
Proof. Since I is an ideal of X, we have that for all x, $y \in X$, if $x \in I$ and $x * y \in I$ then $y \in J$. Now, since $J \subseteq X$, we have that for all x, $y \in J$, if $x \in I$ and $x * y \in I$ then $y \in I$. Therefore I is an ideal in J.

Lemma 3.15 Let I be a closed ideal of a BBG-algebra X. For all a, $b \in X$, Then:

(1) $[a]_I = I$ iff $a \in I$.

(2) Either $[a]_I = [b]_I$ or $[a]_I \cap [b]_I = \varnothing$.

Proof. Let I be a closed ideal of a BBG-algebra X and a, $b \in X$.
(1) We first prove that if $[a]_I = I$, then $a \in I$. By definition, $x \in [a]_I$ if and only if $x \in X$ and $x \sim a$, i.e., $x * a \in I$ and $a * x \in I$. Since $a \sim a$ and $a * a = 0 \in I$, we have $a \in [a]_I$. Since $[a]_I = I$, we have $a \in I$. We prove that if $a \in I$, then $[a]_I = I$ by proving that if $x \in I$ then $x \in [a]_I$ and if $x \in [a]_I$ then $x \in I$. Let $x \in I$. Then, since $a \in I$ and I is a closed ideal, we have $a * x \in I$ and $x * a \in I$. Therefore, a $\sim$ x and hence $x \in [a]_I$. Let $x \in [a]_I$. Then, $a * x \in I$ and $x * a \in I$. Since $a \in I$ and $a * x \in I$, we have $x \in I$ from the definition of the ideal I.
(2) We consider three cases: For a closed ideal I either (a) a, $b \in I$, (b) $a \in I$ and $b \notin I$ and (c) a, $b \notin I$.

(a) From part 1, if a, $b \in I$, then $[a]_I = I$ and $[b]_I = I$. Therefore, $[a]_I = [b]_I$.

(b) Assume that $a \in I$, $b \notin I$ and that $[a]_I \cap [b]_I$ is not an empty set. There exists an element $x \in X$ such that $x \in [a]_I$ and $x \in [b]_I$. However, by definition of $[b]_I$ if $x \in [b]_I$, then $x * b \in I$ and $b * x \in I$. But, from part 1, if $x \in [a]_I$, then $x \in I$. Therefore, $x \in I$ and $x * b \in I$. By definition of the ideal I, we have the contradiction that $b \in I$.

(c) From part 1, if $a,b \notin I$, then $[a]_I$, $[b]_I = I$. Therefore, $[a]_I = [b]_I$ or $[a]_I \cap [b]_I = \varnothing$.

Corollary 3.16 If I is a closed ideal of a BBG-algebra X and $y \in I$, then $[y]_I$ is a closed ideal of X.
Proof. From part 1 of Lemma 3.15, if I is a closed ideal of X and $y \in I$, then $[y]_I = I$ is also closed.

Example 3.17 Let $X = \{0, 1, 2, 3\}$ be a commutative BBG-algebra with $*$ operation defined in Table 3 of Example 2.5 and $I = \{0, 2\}$ is closed ideals of X. Then the quotient set as $X/I = \{ [0]_I, [1]_I \}$, where $[0]_I = [2]_I = \{ 0, 2 \}$ and $[1]_I = [3]_I = \{1, 3\}$. From Lemma 3.15 we will shown that,

1. $[a]_I = I$ iff $a \in I$, we will have $[0]_I = [2]_I = \{ 0, 2 \} = I$ and $0, 2 \in I$.
2. $[a]_I = [b]_I$ or $[a]_I \cap [b]_I = \varnothing$.
   (a) For a, $b \in I$, we have $0, 2 \in I$ and that $[0]_I = [2]_I$.
   (b) For $a \in I$ and $b \notin I$, we have $0 \in I$ and $1 \notin I$, such that $[0]_I \cap [1]_I = \varnothing$.
   (c) For a, $b \notin I$, we have $1, 3 \notin I$ and that $[1]_I = [3]_I$.

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